

# A NOTE ON STOCHASTIC NAVIER-STOKES EQUATIONS WITH NOT REGULAR MULTIPLICATIVE NOISE

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*Dedicated to Professor Bolesław Szafirski on his 80th birthday.*

ABSTRACT. We consider the Navier-Stokes equations in  $\mathbb{R}^d$  ( $d = 2, 3$ ) with a stochastic forcing term which is white noise in time and coloured in space; the spatial covariance of the noise is not too regular, so Itô calculus cannot be applied in the space of finite energy vector fields. We prove existence of weak solutions for  $d = 2, 3$  and pathwise uniqueness for  $d = 2$ .

## 1. INTRODUCTION

The aim of this paper is to study the stochastic Navier-Stokes equations with multiplicative noise, that is the equations of motion of a viscous incompressible fluid with two forcing terms, one is deterministic and the other one is random. The equations are

$$(1.1) \quad \begin{cases} \partial_t v + [-\nu \Delta v + (v \cdot \nabla)v + \nabla p] dt = G(v)dw + f dt \\ \nabla \cdot v = 0 \end{cases}$$

where the unknowns are the velocity  $v = v(t, x)$  and the pressure  $p = p(t, x)$ . By  $\nu > 0$  we denote the viscosity coefficient and in our model the stochastic force can depend on the velocity itself. We consider  $x \in \mathbb{R}^d$  for  $d = 2, 3$  and  $t \geq 0$ . The above equations are associated with an initial condition

$$(1.2) \quad v(0, x) = v_0(x)$$

where  $v_0$  is a divergence free square integrable vector field on  $\mathbb{R}^d$ .

The problem of existence and uniqueness of solutions on any time interval has been already studied in the case of a unbounded domain in [7], [6] and in the case of full euclidean space  $\mathbb{R}^d$  in [30], [24], [25], always with smooth assumptions on the noise term. The problem in bounded domains has been considered in many papers, with different assumptions on the noise. From the technical point of view, it is easier to work in a bounded domain, since the issue of compactness needed in the proof of existence is more involved in unbounded spatial domains.

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Inspired by [16] (Section 3.4), we consider the Navier-Stokes equations (1.1) with a multiplicative noise whose covariance is not regular enough to allow to use Itô formula in the space of finite energy velocity vectors, which is the basic space in which one looks for existence of solutions. In this way the noise is rougher than in [7], [30], [24], [25]. Our original aim was to investigate the existence of invariant measures and stationary solutions for these stochastic Navier-Stokes equations, following on one side the work of [16] in bounded domains and on the other side the method by the first named authour with Motyl and Ondreját [9] for unbounded domains but with more regular noise, which is based on [23]. While working on this problem we realised that the existence or uniqueness of solutions was left open in some cases; indeed, [16] proved existence of martingale and stationary martingale solutions with rough multiplicative noise only for  $d = 2$ .

The aim of this note is to prove existence of a martingale solution for the Navier-Stokes equations (1.1) in  $\mathbb{R}^d$  for  $d = 2$  as well as  $d = 3$ , when the covariance of the noise is not a trace class operator in the basic space  $H$  of finite energy. When  $d = 2$  we prove pathwise uniqueness; this implies existence of a strong solution too. We point out that working in Banach spaces (and using  $\gamma$ -radonifying operators instead of Hilbert-Schmidt operators) we can make assumptions on the covariance of the noise, which are weaker than in [16] to get existence of martingale solutions. However, the argument to prove existence of martingale solutions comes from Section 3.4 of [16]. The original problem on invariant measures in unbounded domains will be studied in a subsequent paper.

As far as the content of the paper are concerned, in Section 2 we define the abstract setting in order to write (1.1) as an Itô equation in some Banach space. Then Section 3 deals with the existence result, whereas uniqueness when  $d = 2$  is proved in Section 4. Definition and properties of  $\gamma$ -radonifying operators and some compactness lemmas are given in the Appendix.

## 2. MATHEMATICAL FRAMEWORK

We first introduce the functional spaces.

For  $1 \leq p < \infty$  let  $L^p = [L^p(\mathbb{R}^d)]^d$  with norm

$$\|v\|_{L^p} = \left( \sum_{k=1}^d \|v^k\|_{L^p(\mathbb{R}^d)}^p \right)^{\frac{1}{p}}$$

where  $v = (v^1, \dots, v^d)$ .

Set  $J^s = (I - \Delta)^{\frac{s}{2}}$ . We define the generalized Sobolev spaces of divergence free vector distributions as

$$(2.1) \quad H^{s,p} = \{u \in [\mathcal{S}'(\mathbb{R}^d)]^d : \|J^s u\|_{L^p} < \infty\},$$

$$(2.2) \quad H_{\text{sol}}^{s,p} = \{u \in H^{s,p} : \nabla \cdot u = 0\}$$

for  $s \in \mathbb{R}$  and  $1 \leq p \leq \infty$ . The divergence has to be understood in the weak sense. We have, see [2], that  $J^\sigma$  is an isomorphism between  $H^{s,p}$  and  $H^{s-\sigma,p}$ . Moreover  $H^{s_2,p} \subset H^{s_1,p}$  when  $s_1 < s_2$  and the dual space of  $H^{s,p}$  is the space  $H_q^{-s}$  with  $1 < q \leq \infty$ :  $\frac{1}{p} + \frac{1}{q} = 1$ . we denote by  $\langle \cdot, \cdot \rangle$  the  $H^{s,p} - H^{-s,q}$  duality bracket:

$$\langle u, v \rangle = \sum_{k=1}^d \int_{\mathbb{R}^d} (J^s u^k)(x) (J^{-s} v^k)(x) dx.$$

In particular, for the Hilbert case  $p = 2$  we set  $H = H_{\text{sol}}^{0,2}$  and, for  $s \neq 0$ ,  $H^s = H_{\text{sol}}^{s,2}$ ; that is

$$H = \{v \in [L^2(\mathbb{R}^d)]^d : \nabla \cdot v = 0\}$$

with scalar product inherited from  $[L^2(\mathbb{R}^d)]^d$ .

We recall the Sobolev embedding theorem, see, e.g., [2, Th. 6.5.1]. If  $1 < q < p < \infty$  with

$$\frac{1}{p} = \frac{1}{q} - \frac{r-s}{d}$$

then the following inclusion holds

$$H^{r,q} \subset H^{s,p}$$

and there exists a constant  $C$  (depending on  $r-s, p, q, d$ ) such that

$$\|v\|_{H^{s,p}} \leq C \|v\|_{H^{r,q}} \quad \text{for all } v \in [\mathcal{S}'(\mathbb{R}^d)]^d.$$

By Lemma 2.5 of [17], see also Lemma C.1 in [7], there exists a separable Hilbert space  $U$  such that  $U$  is a dense subset of  $H^1$  and is compactly embedded in  $H^1$ . We also have that

$$U \subset H^1 \subset H \simeq H' \subset H^{-1} \subset U'$$

with dense and continuous embeddings, but in addition  $H^{-1}$  is compactly embedded in  $U'$ .

Now we define the operators appearing in the abstract formulation of (1.1). We refer to [21] and [29] for the details.

Let  $A = -\Pi\Delta$ , where  $\Pi$  is the projector onto the space of divergence free vector fields. Then  $A$  is a linear unbounded operator in  $H^{s,p}$  as well as in  $H_{\text{sol}}^{s,p}$  ( $s \in \mathbb{R}$ ,  $1 \leq p < \infty$ ), which generates a contractive and analytic  $C_0$ -semigroup  $\{e^{-tA}\}_{t \geq 0}$ . Moreover, for  $t > 0$  the operator  $e^{-tA}$  is bounded from  $H_{\text{sol}}^{s,p}$  into  $H_{\text{sol}}^{s',p}$  with  $s' > s$  and there exists a constant  $M$  (depending on  $s' - s$  and  $p$ ) such that, see Lemma 1.2 in the Kato-Ponce paper [21],

$$(2.3) \quad \|e^{-tA}v\|_{H_{\text{sol}}^{s',p}} \leq M(1 + t^{-(s'-s)/2}) \|v\|_{H_{\text{sol}}^{s,p}}$$

i.e.

$$(2.4) \quad \|e^{-tA}\|_{\mathcal{L}(H_{\text{sol}}^{s,p}, H_{\text{sol}}^{s',p})} \leq M(1 + t^{-(s'-s)/2}).$$

We set

$$\|\nabla v\|_{L^2}^2 = \sum_{k=1}^d \|\nabla v^k\|_{L^2}^2, \quad v \in H^1.$$

We have  $A : H^1 \rightarrow H^{-1}$  and

$$\langle Av, v \rangle = \|\nabla v\|_{L^2}^2, \quad v \in H^1.$$

Moreover

$$(2.5) \quad \|v\|_{H^1}^2 = \|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2$$

We define the bilinear operator  $B : H^1 \times H^1 \rightarrow H^{-1}$  as

$$\langle B(u, v), z \rangle = \int_{\mathbb{R}^d} (u(x) \cdot \nabla)v(x) \cdot z(x) \, dx.$$

The form  $B$  is bounded, since by Hölder and Sobolev inequalities

$$(2.6) \quad |\langle B(u, v), z \rangle| \leq \|u\|_{L^4} \|\nabla v\|_{L^2} \|v\|_{L^4} \leq C \|u\|_{H^1} \|v\|_{H^1} \|z\|_{H^1}.$$

Moreover, see e.g. [29],

$$(2.7) \quad \langle B(u, v), z \rangle = -\langle B(u, z), v \rangle, \quad \langle B(u, v), v \rangle = 0.$$

Using (2.6)-(2.7) and the fact that  $H^1$  is dense in  $L^4$ ,  $B$  can be extended to a bounded bilinear operator from  $L^4 \times L^4$  to  $H^{-1}$  and

$$(2.8) \quad \|B(u, v)\|_{H^{-1}} \leq \|u\|_{L^4} \|v\|_{L^4}.$$

We shall need an estimate of  $B(u, v)$  in bigger spaces.

**Lemma 2.1.** *Let  $d = 2$  and  $g \in (0, 1)$ . Then there exists a constant  $C_g$  such that for all  $u \in H^{1-g}$ ,  $v \in H^{\frac{1-g}{2}}$ ,*

$$(2.9) \quad \|B(u, v)\|_{H^{-1-g}} + \|B(v, u)\|_{H^{-1-g}} \leq C_g \|u\|_{H^{-g}}^{\frac{1-g}{2}} \|u\|_{H^{1-g}}^{\frac{1+g}{2}} \|v\|_{H^{\frac{1-g}{2}}}.$$

*Proof.* Since the space  $H^{-1-g}$  is dual to  $H^{1+g}$  we have

$$\|B(u, v)\|_{H^{-1-g}} = \sup_{\|\phi\|_{H^{1+g}} \leq 1} |\langle B(u, v), \phi \rangle|.$$

For smooth vectors of compact support, we have

$$(2.10) \quad \begin{aligned} \langle B(u, v), \phi \rangle &= \int_{\mathbb{R}^2} ([u(x) \cdot \nabla] v(x)) \cdot \phi(x) \, dx \\ &= \sum_{i,j=1}^2 \int_{\mathbb{R}^2} \partial_i (u^i(x) v^j(x)) \phi^j(x) \, dx \\ &= - \sum_{i,j=1}^2 \int_{\mathbb{R}^2} u^i(x) v^j(x) \partial_i \phi^j(x) \, dx \end{aligned}$$

and similarly

$$(2.11) \quad \langle B(v, u), \phi \rangle = - \sum_{i,j=1}^2 \int_{\mathbb{R}^2} v^i(x) u^j(x) \partial_i \phi^j(x) \, dx.$$

This holds also for less regular vectors, by density. For scalar functions, Hölder inequality and Sobolev embedding (in two dimensions) give

$$(2.12) \quad \begin{aligned} \left| \int_{\mathbb{R}^2} f g h \, dx \right| &\leq \|f\|_{L_{p_1}} \|g\|_{L_{p_2}} \|h\|_{L_{p_3}} \\ &\leq C \|f\|_{H^{\frac{1-g}{2}}} \|g\|_{H^{\frac{1-g}{2}}} \|h\|_{H^g} \end{aligned}$$

choosing  $p_1 = p_2 = \frac{4}{1+g}$  and  $p_3 = \frac{2}{1-g}$  so that  $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$ .

Hence, from (2.10)-(2.12) we get

$$\|B(u, v)\|_{H^{-1-g}} + \|B(v, u)\|_{H^{-1-g}} \leq C \|u\|_{H^{\frac{1-g}{2}}} \|v\|_{H^{\frac{1-g}{2}}}.$$

Moreover, by the complex interpolation  $H^{\frac{1-g}{2}} = [H^{-g}, H^{1-g}]_{\frac{1+g}{2}}$  we have

$$\|u\|_{H^{\frac{1-g}{2}}} \leq C \|u\|_{H^{-g}}^{\frac{1-g}{2}} \|u\|_{H^{1-g}}^{\frac{1+g}{2}}.$$

This allows to conclude an estimate for  $\|B(u, v)\|_{H^{-1-g}}$ ; in the same way we deal with  $\|B(v, u)\|_{H^{-1-g}}$ .  $\square$

We notice that in the proof as well in the sequel, we denote by  $C$  different generic constants; if we need to specify it, we label it in a peculiar way.

Finally, we define the noise forcing term. Given a real separable Hilbert space  $Y$  we consider a  $Y$ -cylindrical Wiener process  $w$  defined on a stochastic basis  $(\Omega, \mathbb{F}, \mathbb{P})$  where  $\mathbb{F} = \{\mathbb{F}_t\}_{t \geq 0}$  is a right continuous filtration, see e.g. [13]. This means that

$$w(t) = \sum_{j=1}^{\infty} \beta_j(t) e_j, \quad t \geq 0,$$

where  $\{e_j\}_{j \in \mathbb{N}}$  is a complete orthonormal system in  $Y$  and  $\{\beta_j\}_{j \in \mathbb{N}}$  is a sequence of independent identically distributed standard Wiener processes defined on  $(\Omega, \mathbb{F}, \mathbb{P})$ .

In the Appendix we recall the definition and basic properties of  $\gamma$ -radonifying operators. For the covariance of the noise we make the following assumptions:

**(G1)**  $\exists g \in (0, 1)$  such that the mapping  $G : H \rightarrow \gamma(Y; H^{-g})$  is well defined, continuous and

$$\sup_{v \in H} \|G(v)\|_{\gamma(Y; H^{-g})} =: K_{g,2} < \infty$$

**(G2)**  $\exists g \in (0, 1)$  such that the mapping  $G : H \rightarrow \gamma(Y; H_{\text{sol}}^{-g,4})$  is well defined, continuous and

$$\sup_{v \in H} \|G(v)\|_{\gamma(Y; H_{\text{sol}}^{-g,4})} =: K_{g,4} < \infty$$

**(G3)** If assumption **(G1)** holds, then  $G$  extends to a Lipschitz continuous map  $G : H^{-g} \rightarrow \gamma(Y; H^{-g})$ , i.e.

$$\exists L_g > 0 : \|G(v_1) - G(v_2)\|_{\gamma(Y; H^{-g})} \leq L_g \|v_1 - v_2\|_{H^{-g}} \quad \forall v_1, v_2 \in H^{-g}.$$

**Remark 2.2.** *i) A map  $G : H \rightarrow \gamma(Y; H_{\text{sol}}^{-g,p})$  is well defined iff the map  $J^{-g}G : H \rightarrow \gamma(Y; H_{\text{sol}}^{0,p})$  is well defined. Moreover*

$$\|J^{-g}G(v)\|_{\gamma(Y; H_{\text{sol}}^{0,p})} = \|G(v)\|_{\gamma(Y; H_{\text{sol}}^{-g,p})} \leq K_{g,p}, \quad v \in H.$$

*For  $g \leq 0$ , assumption **(G1)** would mean that  $G(v) : Y \rightarrow H$  is a Hilbert-Schmidt operator, but we make weaker assumptions on  $G$ , that is we consider  $g > 0$ .*

*Notice that if any of the three conditions above holds for some  $g$ , then it holds for any  $g' > g$  too.*

*ii) In the particular case of  $G(v)e_j = \sigma_j(v)e_j$  (with  $\sigma_j : H \rightarrow \mathbb{R}$  and  $Y = H$ ) we have*

$$\begin{aligned} \|G(v)\|_{\gamma(Y; H_{\text{sol}}^{-g,p})}^p &= \|J^{-g}G(v)\|_{\gamma(Y; H_{\text{sol}}^{0,p})}^p \\ &= \int_{\mathbb{R}^d} \left( \sum_{j=1}^{\infty} \sigma_j(v(x))^2 |(J^{-g}e_j)(x)|^2 \right)^{\frac{p}{2}} dx \\ &\leq \int_{\mathbb{R}^d} \left( \sum_j \|\sigma_j\|_{\infty}^2 |(J^{-g}e_j)(x)|^2 \right)^{\frac{p}{2}} dx, \end{aligned}$$

where  $\|\sigma_j\|_{\infty}^2 := \sup_{v \in H} \sigma_j(v)^2$ .

Taking  $p = 2$  we infer that the condition **(G1)** holds if

$$\sum_{j=1}^{\infty} \|\sigma_j\|_{\infty}^2 \|e_j\|_{H^{-g}}^2 < \infty.$$

However, condition **(G1)** is more involved.

iii) According to Proposition 5.2 we have

$$(2.13) \quad \mathbb{E} \left\| \int_0^t G(v(s)) dw(s) \right\|_{H^{-g}}^m \leq C_m (K_{g,2})^m t^{m/2}.$$

Projecting the first equation of (1.1) onto the space of divergence free vector fields, we get rid of the pressure term and we can write the stochastic Navier-Stokes equations (1.1) in abstract form as

$$(2.14) \quad \begin{cases} dv(t) + [Av(t) + B(v(t), v(t))]dt = G(v(t)) dw(t) + f(t) dt, & t \in (0, T] \\ v(0) = v_0 \end{cases}$$

We assume from now on that  $v_0 \in H$  and  $f \in L^p(0, T; H^{-1})$  for some  $p > 2$ , see Assumption **A.2** and Theorem 5.1 in [7]. Now, we denote by  $C([0, T]; H_w)$  the space of  $H$ -valued weakly continuous functions with the topology of uniform weak convergence on  $[0, T]$ ; in particular  $v_n \rightarrow v$  in  $C([0, T]; H_w)$  means

$$\lim_{n \rightarrow \infty} \sup_{0 \leq t \leq T} |(v_n(t) - v(t), h)_H| = 0$$

for all  $h \in H$ . Notice that  $v(t) \in H$  for any  $t$  if  $v \in C([0, T]; H_w)$ .

Our aim is to find a martingale solution to (2.14). By this we mean a weak solution in the probabilistic sense, according to the following

**Definition 2.3** (solution to the martingale problem). *We say that there exists a martingale solution of (2.14) if there exist*

- a stochastic basis  $(\hat{\Omega}, \hat{\mathbb{F}}, \hat{\mathbb{P}})$
- a  $Y$ -cylindrical Wiener process  $\hat{w}$
- a progressively measurable process  $v : [0, T] \times \hat{\Omega} \rightarrow H$  with  $\hat{\mathbb{P}}$ -a.e. path

$$v \in C([0, T]; H_w) \cap L^2(0, T; L^4)$$

and for any  $t \in [0, T], \psi \in H^2$

$$(2.15) \quad \begin{aligned} \langle v(t), \psi \rangle + \int_0^t \langle Av(s), \psi \rangle ds + \int_0^t \langle B(v(s), v(s)), \psi \rangle ds \\ = \langle v_0, \psi \rangle + \int_0^t \langle f(s), \psi \rangle ds + \left\langle \int_0^t G(v(s)) d\hat{w}(s), \psi \right\rangle \end{aligned}$$

$\hat{\mathbb{P}}$ -a.s.

The regularity of the paths of this solution makes all the terms in (2.15) well defined, thanks to (2.8) and (2.13).

## 3. EXISTENCE OF SOLUTIONS

Looking for martingale solutions for system (2.14) one cannot use Itô calculus in the space  $H$ , since the covariance of the noise is not regular enough. Therefore, we introduce an approximating system by regularizing the covariance of the noise; this gives a sequence of approximating processes  $\{v_n\}_n$ . In order to pass to the limit as  $n \rightarrow \infty$  we need the tightness of the sequence of their laws. This is obtained by working with two auxiliary processes  $z_n$  and  $u_n$  with  $v_n = z_n + u_n$  in a similar way to [16].

Therefore, we first introduce a smoother problem which approximates (2.14); then we prove the tightness of the sequence of the laws; finally we show the convergence, providing existence of a martingale solution to (2.14).

**3.1. The approximating equation.** We start by defining the sequences

$$R_n = n(nI + A)^{-1} \quad G_n = R_n G, \quad n = 1, 2, \dots$$

We have, see [26], Sect. 1.3, that each  $R_n$  is a contraction operator in  $H_{\text{sol}}^{s,p}$  and it converges strongly to the identity operator, i.e.

$$\|R_n\|_{\mathcal{L}(H_{\text{sol}}^{s,p}; H_{\text{sol}}^{s,p})} \leq 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} R_n h = h \quad \forall h \in H_{\text{sol}}^{s,p}.$$

Moreover, it is a linear bounded operator from  $H_{\text{sol}}^{s,p}$  to  $H_{\text{sol}}^{s+t,p}$  for any  $t \leq 2$ ; but the operator norm is not uniformly bounded in  $n$  for  $t > 0$ . We point out the case for  $p = 2$ , needed in the sequel; we have

$$\|R_n\|_{\mathcal{L}(H_{\text{sol}}^s; H_{\text{sol}}^{s+t})} = \sup_{\|u\|_{H_{\text{sol}}^s} \leq 1} \|R_n u\|_{H_{\text{sol}}^{s+t}}$$

and denoting by  $\hat{u} = \hat{u}(\xi)$  the Fourier transform of  $u$

$$(3.1) \quad \|R_n u\|_{H_{\text{sol}}^{s+t}} = \|J^t R_n J^s u\|_{L^2} = \|n \frac{(1 + |\xi|^2)^{\frac{t}{2}}}{n + |\xi|^2} (1 + |\xi|^2)^{\frac{s}{2}} \hat{u}(\xi)\|_{L^2} \leq B_n \|u\|_{H_{\text{sol}}^s}$$

with  $B_n := \|n \frac{(1 + |\xi|^2)^{\frac{t}{2}}}{n + |\xi|^2}\|_{L^\infty} = C_t \frac{n}{(n-1)^{1-\frac{t}{2}}}$ , given  $t \in (0, 2)$ . For large  $n$ , the quantity  $B_n$  behaves like  $n^{\frac{t}{2}}$ , which is not bounded.

From the above and Baxendale [1]

$$(3.2) \quad \|G_n(v)\|_{\gamma(Y; H_{\text{sol}}^{-g,p})} \leq \|G(v)\|_{\gamma(Y; H_{\text{sol}}^{-g,p})} \quad \forall n$$

and

$$(3.3) \quad \lim_{n \rightarrow \infty} \|G_n(v) - G(v)\|_{\gamma(Y; H_{\text{sol}}^{-g,p})} = 0.$$

On the other hand, the operator  $G_n(v)$  is more regular than  $G(v)$ . Indeed, assuming **(G1)**,  $G_n(v)$  is a Hilbert-Schmidt operator in  $H$ , i.e.

$$(3.4) \quad \begin{aligned} \|G_n(v)\|_{\gamma(Y; H)} &\leq \|R_n J^g\|_{\mathcal{L}(H; H)} \|J^{-g} G(v)\|_{\gamma(Y; H)} \\ &\leq \|R_n\|_{\mathcal{L}(H; H^g)} \|G(v)\|_{\gamma(Y; H^{-g})}. \end{aligned}$$

For any  $n \in \mathbb{N}$ , we consider the following auxiliary problem

$$(3.5) \quad \begin{cases} dv(t) + [Av(t) + B(v(t), v(t))]dt = G_n(v(t))dw(t) + f(t)dt, & t \in (0, T] \\ v(0) = v_0 \end{cases}$$

This is the Navier-Stokes equation (2.14) with a more regular noise. Thanks to (3.4), the operator  $G_n$  is regular enough and one can prove that there exists a

martingale solution for each  $n$ . The result is obtained by means of Itô formula for  $d\|v_n(t)\|_H^2$ , as in Theorem 5.1 of [7] or Theorem 2.1 of [25]. More precisely

**Proposition 3.1.** *Let  $v_0 \in H$  and  $f \in L^p(0, T; H^{-1})$  for some  $p > 2$ . If the assumption **(G1)** is satisfied, then for each  $n$  there exists a martingale solution  $((\Omega_n, \mathbb{F}_n, \mathbb{P}_n), w_n, v_n)$  of (3.5); in addition, there exists a constant  $C_n$ , depending also on  $T$ ,  $\|f\|_{L^p(0, T; H^{-1})}$ ,  $\sup_{v \in H} \|G_n(v)\|_{\gamma(Y; H)}$ , such that*

$$(3.6) \quad \mathbb{E}_n \left[ \sup_{0 \leq t \leq T} \|v_n(t)\|_H^2 + \int_0^T \|\nabla v_n(t)\|_{L^2}^2 dt \right] \leq C_n.$$

Moreover, if  $d = 2$  then  $v_n \in C([0, T]; H)$   $\mathbb{P}_n$ -a.s.

**3.2. Tightness.** The estimates (3.6) are not uniform with respect to  $n$ , since (3.1) and (3.4) show that the Hilbert-Schmidt norms of the  $G_n(v)$  are not uniformly bounded. Therefore from (3.6) one cannot get the tightness of the sequence of the laws of  $v_n$ . For this reason, in order to find suitable uniform estimates for the sequence  $(u_n)_n$  we will follow a different path, which in fact has been used in [16] as well as other works, see for instance [4]. To be precise, we split our problem in two subproblems in the unknowns  $z_n$  and  $u_n$  with  $v_n = u_n + z_n$ . Given the processes  $v_n$  and  $W_n$  from Proposition 3.1, we define the process  $z_n$  as the solution of the Ornstein-Uhlenbeck equation

$$(3.7) \quad dz_n(t) + Az_n(t) dt = G_n(v_n(t)) dw_n(t), \quad t \in (0, T]; \quad z_n(0) = 0.$$

Therefore the process  $u_n = v_n - z_n$  solves

$$(3.8) \quad \frac{du_n}{dt}(t) + Au_n(t) + B(v_n(t), v_n(t)) = f(t), \quad t \in (0, T]; \quad u_n(0) = v_0$$

We will first analyze the Ornstein-Uhlenbeck processes; each  $z_n$  satisfies the following identity

$$(3.9) \quad z_n(t) = \int_0^t e^{-(t-s)A} G_n(v_n(s)) dw_n(s).$$

We have two regularity results. The first of them shows the difference between our approach and that of [16], as we employ properties of the Itô stochastic integral in the 2-smooth Banach spaces as  $L^4$  and  $H_{\text{sol}}^{\varepsilon, 4}$ . The second result also depends on this different approach as it is based on certain result from [3] established 2-smooth Banach spaces.

**Lemma 3.2.** *Assume conditions **(G1)** and **(G2)**. Take any  $g_0 \in [g, 1)$  and put  $\varepsilon = g_0 - g \geq 0$ . Then for any integer  $m \geq 2$  there exists a constant  $C$  independent of  $n$  (but depending on  $m, T$  and  $g_0$ ) such that*

$$\mathbb{E}_n \|z_n\|_{L^m(0, T; H_{\text{sol}}^{\varepsilon, 4})}^m \leq (K_{g, 4} M)^m C.$$

In particular  $z_n \in L^m(0, T; H_{\text{sol}}^{0, 4})$   $\mathbb{P}_n$ -a.s.

*Proof.* From Proposition 3.1, for any  $s \in [0, T]$  we have  $v_n(s) \in H$ . We will use Proposition 5.2 and the fact that **(G2)** and (3.2) imply that  $J^{-g} G_n(v_n(s)) \in \gamma(Y; H_{\text{sol}}^{0, 4})$ , for any  $s \in [0, T]$ . Moreover, since  $J^\varepsilon = J^{g_0} J^{\varepsilon - g_0} = J^{g_0} J^{-g}$ , by [3] we



get

$$\begin{aligned}
\mathbb{E}_n \|z_n(t)\|_{H_{\text{sol}}^{\varepsilon,4}}^m &= \mathbb{E}_n \|J^\varepsilon z_n(t)\|_{H_{\text{sol}}^{0,4}}^m \\
&\leq C_m \mathbb{E}_n \left( \int_0^t \|J^\varepsilon e^{-(t-s)A} G_n(v_n(s))\|_{\gamma(Y; H_{\text{sol}}^{0,4})}^2 ds \right)^{m/2} \\
&\leq C_m \mathbb{E}_n \left( \int_0^t \|J^{g_0} e^{-(t-s)A}\|_{\mathcal{L}(H_{\text{sol}}^{0,4}; H_{\text{sol}}^{0,4})}^2 \|J^{-g} G_n(v_n(s))\|_{\gamma(Y; H_{\text{sol}}^{0,4})}^2 ds \right)^{m/2} \\
&\leq C_m (K_{g,4})^m \left( \int_0^t [M^2 + \frac{M^2}{(t-s)^{g_0}}] ds \right)^{m/2} \quad \text{by (2.4)} \\
&= C_m (K_{g,4} M)^m \left( t + \frac{1}{1-g_0} t^{1-g_0} \right)^{\frac{m}{2}} \quad \text{since } g_0 < 1.
\end{aligned}$$

Integrating in time we get the result.  $\square$

For  $0 < \beta < 1$  let  $C^\beta([0, T]; H^\delta)$  be the Banach space of  $H^\delta$ -valued  $\beta$ -Hölder continuous functions endowed with the following norm

$$\|z\|_{C^\beta([0, T]; H^\delta)} = \sup_{0 \leq t \leq T} \|z(t)\|_{H^\delta} + \sup_{0 \leq t < s \leq T} \frac{\|z(t) - z(s)\|_{H^\delta}}{|t - s|^\beta}.$$

**Lemma 3.3.** *Assume (G1) and let*

$$(3.10) \quad 0 \leq \beta < \frac{1-g}{2}.$$

*Then, for any  $p \geq 2$  and  $\delta \geq 0$  such that*

$$(3.11) \quad \beta + \frac{\delta}{2} + \frac{1}{p} < \frac{1-g}{2}$$

*there exists a modification  $\tilde{z}_n$  of  $z_n$  such that*

$$(3.12) \quad \mathbb{E}_n \|\tilde{z}_n\|_{C^\beta([0, T]; H^\delta)}^p \leq \tilde{C}$$

*for some constant  $\tilde{C}$  independent of  $n$  (but depending on  $T$ ,  $\beta$ ,  $\delta$  and  $p$ ).*

*Proof.* We can see this result as a special case of Corollary 3.5 of [3]. Let us fix  $\beta$ ,  $p \geq 2$  and  $\delta \geq 0$  as in conditions (3.10) and (3.11). Then there exists a modification  $\tilde{z}_n$  of  $z_n$  such that

$$(3.13) \quad \mathbb{E}_n \|\tilde{z}_n\|_{C^\beta([0, T]; H^\delta)}^p \leq C \mathbb{E}_n \int_0^T \|G_n(v_n(s))\|_{\gamma(Y; H^{-g})}^p ds.$$

Using (3.2) and assumption (G1) we conclude the proof of (3.12).  $\square$

In what follow we will collect some easy consequences of the previous two fundamental results.

Firstly, taking  $\delta = 0$ ,  $\beta < \frac{1-g}{2}$  and  $p$  big enough in (3.13), we get

$$(3.14) \quad \mathbb{E}_n \|z_n\|_{C^\beta([0, T]; H)} \leq \left( \mathbb{E}_n \|z_n\|_{C^\beta([0, T]; H)}^p \right)^{\frac{1}{p}} \leq (CT)^{\frac{1}{p}} K_{g,2}.$$

Secondly, taking  $\delta = \frac{1-g}{2}$ ,  $\beta = 0$  and  $p$  big enough in (3.13), we get

$$(3.15) \quad \mathbb{E}_n \|z_n\|_{C([0, T]; H^{\frac{1-g}{2}})} \leq \left( \mathbb{E}_n \|z_n\|_{C([0, T]; H^{\frac{1-g}{2}})}^p \right)^{\frac{1}{p}} \leq (CT)^{\frac{1}{p}} K_{g,2}.$$

Hence, a consequence of the above two lemmas is that there exist finite constants  $E_m$ ,  $E_g$  and  $E_{\beta,\delta}$  such that

$$\sup_n \mathbb{E}_n \|z_n\|_{L^m(0,T;H_{\text{sol}}^{0,4})}^m = (E_m)^m$$

$$\sup_n \mathbb{E}_n \|z_n\|_{C([0,T];H^{\frac{1-q}{2}})} = E_g$$

and

$$\sup_n \mathbb{E}_n \|z_n\|_{C^\beta([0,T];H^\delta)}^p = (E_{\beta,\delta})^p$$

where  $\beta$  and  $\delta$  are as in Lemma 3.3.

Therefore, by the Chebyshev inequality we infer that for any given  $\eta > 0$

$$(3.16) \quad \sup_n \mathbb{P}_n(\|z_n\|_{L^m(0,T;H_{\text{sol}}^{0,4})} > \eta) \leq \frac{E_m}{\eta},$$

$$(3.17) \quad \sup_n \mathbb{P}_n(\|z_n\|_{C([0,T];H^{\frac{1-q}{2}})} > \eta) \leq \frac{E_g}{\eta},$$

$$(3.18) \quad \sup_n \mathbb{P}_n(\|z_n\|_{C^\beta([0,T];H^\delta)} > \eta) \leq \frac{E_{\beta,\delta}}{\eta}.$$

The last three inequalities allow us to get uniform estimates in probability for the sequence  $u_n$ , see Proposition 3.4, and consequently, for  $v_n = z_n + u_n$ , see Proposition 3.5.

We recall that each  $u_n$  solves a deterministic equation with two forcing terms: one is random and the other is deterministic. Indeed  $u_n$  solves

$$(3.19) \quad \frac{du_n}{dt}(t) + Au_n(t) = -B(v_n(t), v_n(t)) + f(t)$$

with  $u_n(0) = v_0$ . We analyze the above equation (3.19) pathwise.

We have the following fundamental results about the uniform estimates of the approximating processes  $(u_n)$ .

**Proposition 3.4.** *Assume (G1) and (G2). Let  $v_0 \in H$  and  $f \in L^p(0, T; H^{-1})$  for some  $p > 2$ .*

*Then, for any  $n$  the paths of the process  $u_n = v_n - z_n$  solving (3.19) are such that*

$$u_n \in L^\infty(0, T; H) \cap L^2(0, T; H^1) \cap L^{\frac{8}{3}}(0, T; L^4) \cap C^{1-\frac{4}{3}}([0, T]; H^{-1})$$

*$\mathbb{P}_n$ -a.s., and for any  $\varepsilon > 0$  there exist positive constants  $C_i = C_i(\varepsilon)$  ( $i = 6, \dots, 10$ ) such that*

$$(3.20) \quad \sup_n \mathbb{P}_n(\|u_n\|_{L^\infty(0,T;H)} > C_6) \leq \varepsilon$$

$$(3.21) \quad \sup_n \mathbb{P}_n(\|u_n\|_{L^2(0,T;H^1)} > C_7) \leq \varepsilon$$

$$(3.22) \quad \sup_n \mathbb{P}_n(\|u_n\|_{L^{\frac{4}{1-q}}(0,T;H^{\frac{1-q}{2}})} > C_8) \leq \varepsilon$$

$$(3.23) \quad \sup_n \mathbb{P}_n(\|u_n\|_{L^{\frac{8}{3}}(0,T;L^4)} > C_9) \leq \varepsilon$$

$$(3.24) \quad \sup_n \mathbb{P}_n(\|u_n\|_{C^{1-\frac{4}{3}}([0,T];H^{-1})} > C_{10}) \leq \varepsilon$$

*Proof.* By definition and merging the regularity of  $v_n$  and  $z_n$  we have that  $u_n = v_n - z_n \in C([0, T]; H_w) \mathbb{P}_n$ -a.s.

Using (2.8) and the Gagliardo-Nirenberg inequality, for  $d = 2$  or  $d = 3$ ,

$$(3.25) \quad \|v\|_{L^4} \leq C \|v\|_{L^2}^{1-\frac{d}{4}} \|\nabla v\|_{L^2}^{\frac{d}{4}}$$

we get

$$(3.26) \quad \int_0^T \|B(v_n(s), v_n(s))\|_{H^{-1}}^{\frac{4}{d}} ds \leq \int_0^T \|v_n(s)\|_{L^4}^{\frac{8}{d}} ds$$

$$(3.27) \quad \leq C \|v_n\|_{L^\infty(0, T; L^2)}^{\frac{8}{d}-2} \int_0^T \|\nabla v_n(s)\|_{L^2}^2 ds.$$

This means that (pathwise) the r.h.s. of equation (3.19) belongs to  $L^{\frac{4}{d}}(0, T; H^{-1})$ .

Now, let us to prove that  $u_n \in L^{\frac{4}{d}}(0, T; H^1)$ . For  $d = 2$ , it is a classical result for the Stokes equation, see [29]. For  $d = 3$  let us use classical maximal regularity theory for linear parabolic equations, see e.g. [14]. We write  $u_n$  as a mild solution

$$u_n(t) = e^{-tA} v_0 + \int_0^t e^{-(t-s)A} [f(s) - B(v_n(s), v_n(s))] ds =: e^{-tA} v_0 + g_n(t), \quad t \in [0, T].$$

Then, using the result from [14] we have the following. If  $f - B(v_n, v_n) \in L^{\frac{4}{d}}(0, T; H^{-1})$ , then  $g_n \in L^{\frac{4}{d}}(0, T; H^1)$ . Moreover, since by (2.4),  $\|e^{-tA} v_0\|_{H^1} \leq M(1 + \frac{1}{t^{1/2}}) \|v_0\|_H$  for  $t > 0$ , we infer that for any  $p < 2$ ,  $e^{-\cdot A} v_0 \in L^p(0, T; H^1)$ . Hence, for  $d = 3$  we get  $u_n \in L^{\frac{4}{d}}(0, T; H^1)$ .

Thus, the term  $\langle B(v_n(t), v_n(t)), u_n(t) \rangle$ , appearing in a while, is meaningful for almost every  $t$  thanks to (2.6) and is equal to  $-\langle B(v_n(t), u_n(t)), v_n(t) \rangle$  thanks to (2.7).

Now we look for suitable estimates giving the tightness of the sequence of laws of  $u_n$ . We begin with the usual energy estimate:

$$(3.28) \quad \frac{1}{2} \frac{d}{dt} \|u_n(t)\|_{L^2}^2 + \|\nabla u_n(t)\|_{L^2}^2 = -\langle B(v_n(t), v_n(t)), u_n(t) \rangle + \langle f(t), u_n(t) \rangle$$

We consider the trilinear term; by means of (2.7)-(2.6), the Gagliardo-Nirenberg inequality (3.25) and the Young inequality we get

$$\begin{aligned} -\langle B(v_n, v_n), u_n \rangle &= \langle B(v_n, u_n), v_n \rangle \\ &= \langle B(v_n, u_n), z_n \rangle + \langle B(v_n, u_n), u_n \rangle \\ &= \langle B(v_n, u_n), z_n \rangle \\ &\leq \|v_n\|_{L^4} \|\nabla u_n\|_{L^2} \|z_n\|_{L^4} \\ &\leq \|u_n\|_{L^4} \|\nabla u_n\|_{L^2} \|z_n\|_{L^4} + \|z_n\|_{L^4} \|\nabla u_n\|_{L^2} \|z_n\|_{L^4} \\ &\leq C \|u_n\|_{L^2}^{1-\frac{d}{4}} \|\nabla u_n\|_{L^2}^{1+\frac{d}{4}} \|z_n\|_{L^4} + \|\nabla u_n\|_{L^2} \|z_n\|_{L^4}^2 \\ &\leq \frac{1}{4} \|\nabla u_n\|_{L^2}^2 + C \|u_n\|_{L^2}^2 \|z_n\|_{L^4}^{\frac{8}{4-d}} + C \|z_n\|_{L^4}^4. \end{aligned}$$

Estimating

$$|\langle f, u_n \rangle| \leq \|f\|_{H^{-1}} \|u_n\|_{H^1} \leq \frac{1}{4} \|u_n\|_{H^1}^2 + C \|f\|_{H^{-1}}^2 = \frac{1}{4} \|u_n\|_{L^2}^2 + \frac{1}{4} \|\nabla u_n\|_{L^2}^2 + C \|f\|_{H^{-1}}^2$$

from (3.28) we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u_n(t)\|_{L^2}^2 + \|\nabla u_n(t)\|_{L^2}^2 &\leq \frac{1}{2} \|\nabla u_n(t)\|_{L^2}^2 \\ &+ \frac{1}{4} \|u_n(t)\|_{L^2}^2 + C \|z_n(t)\|_{L^4}^{\frac{8}{4-d}} \|u_n(t)\|_{L^2}^2 + C \|z_n(t)\|_{L^4}^4 + C \|f(t)\|_{H^{-1}}^2 \end{aligned}$$

i.e.

$$(3.29) \quad \frac{d}{dt} \|u_n(t)\|_{L^2}^2 + \|\nabla u_n(t)\|_{L^2}^2 \leq \phi_n(t) \|u_n(t)\|_{L^2}^2 + \psi_n(t)$$

with

$$\phi_n(t) = \frac{1}{2} + 2C \|z_n(t)\|_{L^4}^{\frac{8}{4-d}} \text{ and } \psi_n(t) = 2C \|z_n(t)\|_{L^4}^4 + 2C \|f(t)\|_{H^{-1}}^2, \quad t \in [0, T],$$

By Lemma 3.2, we have  $\phi_n, \psi_n \in L^1(0, T)$  uniformly in  $n$ . Hence, from Gronwall Lemma applied to inequality

$$(3.30) \quad \frac{d}{dt} \|u_n(t)\|_{L^2}^2 \leq \phi_n(t) \|u_n(t)\|_{L^2}^2 + \psi_n(t)$$

we infer that there exist constants  $C_1, C_2 > 0$  such that for all  $n$

$$\begin{aligned} (3.31) \quad \sup_{0 \leq t \leq T} \|u_n(t)\|_{L^2}^2 &\leq \|v_0\|_H^2 e^{\int_0^T \phi_n(r) dr} + \int_0^T e^{\int_s^T \phi_n(r) dr} \psi_n(s) ds \\ &\leq \|v_0\|_H^2 e^{\int_0^T \phi_n(r) dr} + e^{\int_0^T \phi_n(r) dr} \int_0^T \psi_n(s) ds \\ &\leq C_1 \|v_0\|_H^2 + C_2. \end{aligned}$$

Next, integrating inequality (3.29) in time we get

$$\begin{aligned} (3.32) \quad \int_0^T \|\nabla u_n(t)\|_{L^2}^2 dt &\leq \|v_0\|_H^2 + \int_0^T (\phi_n(t) \|u_n(t)\|_{L^2}^2 + \psi_n(t)) dt \\ &\leq \|v_0\|_H^2 + \left( \sup_{0 \leq t \leq T} \|u_n(t)\|_{L^2}^2 \right) \int_0^T \phi_n(t) dt + \int_0^T \psi_n(t) dt \\ &\leq C_3 \|v_0\|_H^2 + C_4. \end{aligned}$$

Bearing in mind inequality (3.16), we infer that for any  $\varepsilon > 0$  there exist constants  $\eta_1, \eta_2 > 0$  such that

$$\sup_n \mathbb{P}(\int_0^T \phi_n(t) dt > \eta_1) \leq \varepsilon, \quad \sup_n \mathbb{P}(\int_0^T \psi_n(t) dt > \eta_2) \leq \varepsilon.$$

Therefore, from (3.31)-(3.32) we get that for any  $\varepsilon > 0$  there exist suitable constants  $R_1, R_2 > 0$  such that

$$\sup_n \mathbb{P}(\|u_n\|_{L^\infty(0, T; H)} > R_1) \leq \varepsilon \quad \sup_n \mathbb{P}(\|\nabla u_n\|_{L^2(0, T; L^2)} > R_2) \leq \varepsilon.$$

This proves (3.20).

The last two inequalities also imply that for any  $\varepsilon > 0$  there exists a suitable constant  $R_3$  such that

$$\sup_n \mathbb{P}(\|u_n\|_{L^2(0, T; H^1)} > R_3) \leq \varepsilon.$$

This proves (3.21).

Since  $H^{\frac{1-g}{2}} = [H, H^1]_{\frac{1-g}{2}}$ , by the Hölder inequality and the properties of the complex interpolation we infer that  $\|u_n\|_{L^{\frac{4}{1-g}}(0,T;H^{\frac{1-g}{2}})} \leq C \|u_n\|_{L^\infty(0,T;H)}^{\frac{1+g}{2}} \|u_n\|_{L^2(0,T;H^1)}^{\frac{1-g}{2}}$ . Therefore for any  $\varepsilon > 0$  there exists a suitable constant  $R_4$  such that

$$\sup_n \mathbb{P}(\|u_n\|_{L^{\frac{4}{1-g}}(0,T;H^{\frac{1-g}{2}})} > R_4) \leq \varepsilon.$$

This proves (3.22).

By means of the Gagliardo-Nirenberg inequality (3.25) we also deduce that for any  $\varepsilon > 0$  there exists a suitable constant  $R_5 > 0$  such that

$$\sup_n \mathbb{P}(\|u_n\|_{L^{\frac{8}{3}}(0,T;L^4)} > R_5) \leq \varepsilon.$$

This proves (3.23).

From equation (3.19) for  $u_n$  we infer that

$$\begin{aligned} \left\| \frac{du_n}{dt} \right\|_{L^{\frac{4}{3}}(0,T;H^{-1})} &\leq \|Au_n\|_{L^{\frac{4}{3}}(0,T;H^{-1})} + \|B(v_n, v_n)\|_{L^{\frac{4}{3}}(0,T;H^{-1})} + \|f\|_{L^{\frac{4}{3}}(0,T;H^{-1})} \\ &\leq \|u_n\|_{L^{\frac{4}{3}}(0,T;H^1)} + \|v_n\|_{L^{\frac{8}{3}}(0,T;L^4)}^2 + \|f\|_{L^{\frac{4}{3}}(0,T;H^{-1})} \quad \text{using (3.26)} \\ &\leq C\|u_n\|_{L^2(0,T;H^1)} + 2\|u_n\|_{L^{\frac{8}{3}}(0,T;L^4)}^2 + 2\|z_n\|_{L^{\frac{8}{3}}(0,T;L^4)}^2 + C\|f\|_{L^2(0,T;H^{-1})}. \end{aligned}$$

Using (3.20), (3.23) and (3.16), we find that for any  $\varepsilon > 0$  there exists a suitable constant  $R_6 > 0$  such that

$$\sup_n \mathbb{P}(\left\| \frac{du_n}{dt} \right\|_{L^{\frac{4}{3}}(0,T;H^{-1})} > R_6) \leq \varepsilon.$$

By the Sobolev embedding theorem,  $H^{1,\frac{4}{3}}(0,T) = \{u \in L^{\frac{4}{3}}(0,T) : u' \in L^{\frac{4}{3}}(0,T)\} \subset C^{1-\frac{4}{3}}([0,T])$ . Hence we infer that there exists a constant  $R_7 > 0$  such that

$$\sup_n \mathbb{P}(\|u_n\|_{C^{1-\frac{4}{3}}([0,T];H^{-1})} > R_7) \leq \varepsilon.$$

This proves (3.24). This completes the proof of Proposition 3.4.  $\square$

Now we need to apply a tightness argument in order to pass to the limit. Merging the estimates (3.16)-(3.18) for the process  $z_n$  and those for  $u_n$  from Proposition 3.4 we get estimates for  $v_n = z_n + u_n$ . These estimates in probability are uniform with respect to  $n$ .

**Proposition 3.5.** *Assume (G1) and (G2), let  $v_0 \in H$  and  $f \in L^p(0,T;H^{-1})$  for some  $p > 2$  and let  $((\Omega_n, \mathbb{F}_n, \mathbb{P}_n), w_n, v_n)$  be a martingale solution of (3.5) as given in Proposition 3.1.*

*Then there exist  $\gamma, \delta > 0$  such that for any  $\varepsilon > 0$  there exist positive constants  $C_i = C_i(\varepsilon)$  ( $i = 1, \dots, 5$ ) such that*

$$\begin{aligned} \sup_n \mathbb{P}_n(\|v_n\|_{L^\infty(0,T;H)} > C_1) &\leq \varepsilon \\ \sup_n \mathbb{P}_n(\|v_n\|_{L^2(0,T;H^\delta)} > C_2) &\leq \varepsilon \\ \sup_n \mathbb{P}_n(\|v_n\|_{L^{\frac{4}{1-g}}(0,T;H^{\frac{1-g}{2}})} > C_3) &\leq \varepsilon \\ \sup_n \mathbb{P}_n(\|v_n\|_{L^{\frac{8}{3}}(0,T;L^4)} > C_4) &\leq \varepsilon \\ \sup_n \mathbb{P}_n(\|v_n\|_{C^\gamma([0,T];H^{-1})} > C_5) &\leq \varepsilon \end{aligned}$$

Actually,  $\gamma = \min(\beta, 1 - \frac{d}{4})$  with  $\beta$  and  $\delta$  fulfilling (3.11). Therefore  $0 < \gamma < \frac{1}{2}$  and  $0 < \delta < 1$ .

**3.3. Convergence.** The latter result provides the tightness to pass to the limit. We have

**Theorem 3.6.** *Let  $v_0 \in H$  and  $f \in L^p(0, T; H^{-1})$  for some  $p > 2$ . If (G1)-(G2) are satisfied, then there exists a martingale solution  $((\tilde{\Omega}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}}), \tilde{w}, \tilde{v})$  of (2.14); in addition*

$$(3.33) \quad \tilde{v} \in L^{\frac{4}{1-g}}(0, T; H^{\frac{1-g}{2}}) \cap L^{\frac{8}{d}}(0, T; L^4) \quad \tilde{\mathbb{P}} - a.s.$$

Moreover, if  $d = 2$  then  $\tilde{v} \in C([0, T]; H)$   $\tilde{\mathbb{P}}$ -a.s.

*Proof.* One proceeds as in [7].

We fix  $0 < \gamma < \frac{1}{2}$  and  $0 < \delta < 1$  appearing in Proposition 3.5 and define the space

$$Z = L^{\frac{8}{d}}_{\mathbf{w}}(0, T; L^4) \cap C([0, T]; U') \cap L^2(0, T; H_{\text{loc}}) \cap C([0, T]; H_{\mathbf{w}})$$

with the topology  $\mathcal{T}$  given by the supremum of the corresponding topologies. According to Lemma 5.5, Proposition 3.5 provides that the sequence of laws of the processes  $v_n$  is tight in  $Z$ .

By the Jakubowski's generalization of the Skohorokod Theorem to nonmetric spaces, see [7] and [20], there exist a subsequence  $\{v_{n_k}\}_{k=1}^{\infty}$ , a stochastic basis  $(\tilde{\Omega}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}})$ ,  $Z$ -valued Borel measurable variables  $\tilde{v}$  and  $\{\tilde{v}_k\}_{k=1}^{\infty}$  such that for any  $k$  the laws of  $v_{n_k}$  and  $\tilde{v}_k$  are the same and  $\tilde{v}_k$  converges to  $\tilde{v}$   $\tilde{\mathbb{P}}$ -a.s. with the topology  $\mathcal{T}$ .

Since each  $\tilde{v}_k$  has the same law as  $v_{n_k}$ , it is a martingale solution to equation (3.5); therefore each process

$$\tilde{M}_k(t) := \tilde{v}_k(t) - \tilde{v}_k(0) + \int_0^t A\tilde{v}_k(s)ds + \int_0^t B(\tilde{v}_k(s), \tilde{v}_k(s))ds - \int_0^t f(s)ds$$

is a martingale with quadratic variation

$$\langle \langle \tilde{M}_k \rangle \rangle(t) = \int_0^t G_k(\tilde{v}_k(s))G_k(\tilde{v}_k(s))^* ds.$$

It is now classical to show, see e.g. [7], that

$$\langle \tilde{M}_k(t) - \tilde{M}(t), \phi \rangle \rightarrow 0$$

for any  $\phi \in H^2$  and every  $t \in [0, T]$ , where

$$\tilde{M}(t) = \tilde{v}(t) - \tilde{v}(0) + \int_0^t A\tilde{v}(s)ds + \int_0^t B(\tilde{v}(s), \tilde{v}(s))ds - \int_0^t f(s)ds.$$

There is no convergence of the quadratic variation processes, but the quadratic variation of the more regular process  $J^{-g}\tilde{M}_k$

$$\langle \langle J^{-g}\tilde{M}_k \rangle \rangle(t) = \int_0^t J^{-g}G_k(\tilde{v}_k(s))G_k(\tilde{v}_k(s))^* J^{-g}ds,$$

is finite and thanks to (3.3) it converges to

$$\int_0^t J^{-g}G(\tilde{v}(s))G(\tilde{v}(s))^* J^{-g}ds$$

as  $n \rightarrow \infty$ .

With usual martingale representation theorem, see e.g. [13] (and newer approaches to this result: [10] and [5]) we can conclude that there exists a  $Y$ -cylindrical Wiener process  $\tilde{w}$  such that

$$J^{-g}\tilde{M}(t) = \int_0^t J^{-g}G(\tilde{v}(s)) d\tilde{w}(s).$$

Therefore  $\tilde{v}$  is a martingale solution to (2.14).

Finally, (3.33) comes from the uniform estimates of Proposition 3.5.  $\square$

**Remark 3.7.** *Our techniques, although primarily devised in order to treat unbounded domains, can be applied for bounded domains as well. So, let us compare our assumptions with that of the seminal paper [16] by Flandoli and Gatarek, where only smooth and bounded domains are considered.*

*Firstly, let us observe that condition (G2), although involving the Banach space  $H_{\text{sol}}^{-g,4}$ , is on the same level of regularity as condition (G1). Secondly, when working in a box with the periodic boundary conditions the assumptions (G1) and (G2) can be reduced to only one assumption according to Remark 2.2 ii). Indeed, one can choose  $\{e_j\}$  to be the eigenvalues of the Stokes operator so that the sequence of the  $e_j(x)$  are uniformly bounded in  $x$ , see, e.g., [15].*

*On the other hand, in Section 3.4 of [16] the authors assume that  $d = 2$  and*

$$(3.34) \quad \exists g \in (0, \tfrac{1}{2}) \quad : \quad \text{the map } G : H \rightarrow \gamma(Y; H^{-g}) \\ \text{is well defined, bounded and continuous.}$$

*With these assumptions on the noise, Flandoli and Gatarek proved, see (25) in [16] (but written here with our notations and correcting a misprint since the exponent there need to have a  $+$  instead of a  $-$ ) that for some  $\varepsilon > 0$  and for all  $m \geq 1$  there exists a constant  $C$  (dependent on  $T$ ,  $m$  and  $\varepsilon$  but independent of  $n$ ) such that*

$$\mathbb{E}_n \|z_n\|_{L^m(0,T;H^{\frac{1}{2}+\varepsilon})}^m \leq C$$

*From this, using the continuous embedding  $H^{\frac{1}{2}+\varepsilon} \subset H_{\text{sol}}^{\varepsilon,4}$  in the two dimensional case, they get the mean estimate in the  $L^4(0,T;H_{\text{sol}}^{\varepsilon,4})$ -norm, which is the basic tool in their proof. However, one can obtain this latter estimate as in our Lemma 3.2, assuming only (G1)-(G2) instead of (3.34). Let us repeat here again that the reason for us being able to weaken the assumptions from [16] is the use of Itô integral with values in 2-smooth Banach spaces such as  $L^4$ .*

*Finally, it seems to us that the argument of [16] to get existence of a martingale solution would work also for  $d = 3$  assuming the following stronger version of the previously recalled assumption (3.34),*

$$(3.35) \quad \exists g \in (0, \tfrac{1}{4}) \quad : \quad \text{the map } G : H \rightarrow \gamma(Y; H^{-g}) \\ \text{is well defined, bounded and continuous.}$$

*Indeed, for  $d = 3$  this gives the uniform estimates for the mean value of  $z_n$  in the norm  $L^4(0,T;H^{\frac{3}{4}+\varepsilon})$ . Since  $H^{\frac{3}{4}+\varepsilon} \subset H_{\text{sol}}^{\varepsilon,4}$  when  $d = 3$ , one gets also the estimates in the norm of  $L^4(0,T;H_{\text{sol}}^{\varepsilon,4})$ .*

*Obviously, the last assumption (3.35) is stronger than our condition (G1) also for  $d = 3$ .*

4. PATHWISE UNIQUENESS FOR  $d = 2$ 

For the case  $d = 2$  we will investigate the pathwise uniqueness. This means that given two processes  $v_1, v_2$  solving (2.14) on the same stochastic basis  $(\Omega, \mathbb{F}, \mathbb{P})$  with the same Wiener process  $w$ , initial velocity  $v_0$  and force  $f$ , we have

$$(4.1) \quad \mathbb{P}\{v_1(t) = v_2(t) \text{ for all } t \in [0, T]\} = 1.$$

We are going to prove the following result.

**Theorem 4.1.** *Assume **(G1)**-(**G2**)-(**G3**). For  $d = 2$ , if  $v_0 \in H$  and  $f \in L^p(0, T; H^{-1})$  for some  $p > 2$ , then there is pathwise uniqueness for equation (2.14).*

*Proof.* Set  $V = v_1 - v_2$ ; this difference satisfies

$$dV(t) + [AV(t) + B(v_1(t), v_1(t)) - B(v_2(t), v_2(t))] dt = [G(v_1(t)) - G(v_2(t))] dw(t)$$

with  $V(0) = 0$ ; this equation is equivalent to

$$dV(t) + [AV(t) + B(V(t), v_1(t)) + B(v_2(t), V(t))] dt = [G(v_1(t)) - G(v_2(t))] dw(t).$$

Following an idea from [8], we will use the Itô formula for  $d\left(e^{-\int_0^t \psi(s) ds} \|V(t)\|_{H^{-g}}^2\right)$ , by choosing  $\psi$  as done in [27]:

$$\psi(s) = 1 + L_g^2 + 2\overline{C} \left( \|v_1(s)\|_{H^{\frac{1-g}{2}}}^{\frac{4}{1-g}} + \|v_2(s)\|_{H^{\frac{1-g}{2}}}^{\frac{4}{1-g}} \right)$$

with  $L_g$  the Lipschitz constant given in **(G3)** and  $\overline{C}$  the constant appearing later on in (4.2). We recall that  $v_1, v_2 \in L^{\frac{4}{1-g}}(0, T; H^{\frac{1-g}{2}})$   $\mathbb{P}$ -a.s., so  $\psi \in L^1(0, T)$   $\mathbb{P}$ -a.s.; moreover,  $V \in C([0, T]; H) \subset C([0, T]; H^{-g})$ .

We have

$$d\left(e^{-\int_0^t \psi(s) ds} \|V(t)\|_{H^{-g}}^2\right) = -\psi(t) e^{-\int_0^t \psi(s) ds} \|V(t)\|_{H^{-g}}^2 dt + e^{-\int_0^t \psi(s) ds} d\|V(t)\|_{H^{-g}}^2$$

and the latter differential is well defined and given by

$$\begin{aligned} \frac{1}{2} d\|V(t)\|_{H^{-g}}^2 &= -\|\nabla V(t)\|_{H^{-g}}^2 dt - \langle J^{-g}[B(V(t), v_1(t)) + B(v_2(t), V(t))], J^{-g}V(t) \rangle dt \\ &\quad + \langle J^{-g}[G(v_1(t)) - G(v_2(t))] dw(t), J^{-g}V(t) \rangle + \frac{1}{2} \|G(v_1(t)) - G(v_2(t))\|_{\gamma(Y; H^{-g})}^2 dt \end{aligned}$$

By means of Lemma 2.1 and Young inequality we get that there exists a constant  $\overline{C}$  such that

$$\begin{aligned} (4.2) \quad |\langle J^{-g}B(V, v_1), J^{-g}V \rangle| &= \langle J^{-1-g}B(V, v_1), J^{1-g}V \rangle \\ &\leq \|B(V, v_1)\|_{H^{-1-g}} \|V\|_{H^{1-g}} \\ &\leq C \|V\|_{H^{-g}}^{\frac{1-g}{2}} \|V\|_{H^{1-g}}^{\frac{1+g}{2}} \|v_1\|_{H^{\frac{1-g}{2}}} \|V\|_{H^{1-g}} \\ &= C \|V\|_{H^{-g}}^{\frac{1-g}{2}} \|V\|_{H^{1-g}}^{\frac{3+g}{2}} \|v_1\|_{H^{\frac{1-g}{2}}} \\ &\leq \frac{1}{4} \|V\|_{H^{1-g}}^2 + \overline{C} \|v_1\|_{H^{\frac{1-g}{2}}}^{\frac{4}{1-g}} \|V\|_{H^{-g}}^2 \\ &= \frac{1}{4} \|\nabla V\|_{H^{-g}}^2 + \frac{1}{4} \|V\|_{H^{-g}}^2 + \overline{C} \|v_1\|_{H^{\frac{1-g}{2}}}^{\frac{4}{1-g}} \|V\|_{H^{-g}}^2 \end{aligned}$$

where we used that  $\|V(t)\|_{H^{1-g}}^2 = \|\nabla V(t)\|_{H^{-g}}^2 + \|V(t)\|_{H^{-g}}^2$ , see (2.5)). Similarly we get the estimate for  $\langle J^{-g}B(v_2, V), J^{-g}V \rangle$ .



Therefore, using **(G3)** we get

$$(4.3) \quad d \left( e^{-\int_0^t \psi(s) ds} \|V(t)\|_{H^{-g}}^2 \right) \leq e^{-\int_0^t \psi(s) ds} \langle J^{-g}[G(v_1(t)) - G(v_2(t))] dw(t), J^{-g}V(t) \rangle.$$

The r.h.s. is a local martingale; indeed if we define the stopping time

$$\tau_N = T \wedge \inf\{t \in [0, T] : \|V(t)\|_{H^{-g}} > N\}$$

and

$$M_N(t) = \int_0^{t \wedge \tau_N} e^{-\int_0^r \psi(s) ds} \langle J^{-g}V(r), J^{-g}[G(v_1(r)) - G(v_2(r))] dw(r) \rangle$$

then

$$\begin{aligned} \mathbb{E}M_N(t)^2 &\leq \mathbb{E} \int_0^{t \wedge \tau_N} e^{-2\int_0^r \psi(s) ds} \|V(r)\|_{H^{-g}}^2 \|G(v_1(r)) - G(v_2(r))\|_{\gamma(Y; H^{-g})}^2 dr \\ &\leq L_g^2 \mathbb{E} \int_0^{t \wedge \tau_N} \|V(r)\|_{H^{-g}}^4 dr \\ &\leq L_g^2 N^4 t. \end{aligned}$$

Hence,  $M_N$  is a square integrable martingale; in particular  $\mathbb{E}M_N(t) = 0$  for any  $t$ .

Therefore, by integrating (4.3) over  $[0, t \wedge \tau_N]$  and taking the expectation we get

$$\mathbb{E} e^{-\int_0^{t \wedge \tau_N} \psi(s) ds} \|V(t \wedge \tau_N)\|_{H^{-g}}^2 \leq 0.$$

So

$$e^{-\int_0^{t \wedge \tau_N} \psi(s) ds} \|V(t \wedge \tau_N)\|_{H^{-g}}^2 = 0 \quad \mathbb{P} - a.s.$$

Since  $\lim_{N \rightarrow \infty} \tau_N = T$   $\mathbb{P}$ -a.s., we get in the limit that for any  $t \in [0, T]$

$$e^{-\int_0^t \psi(s) ds} \|V(t)\|_{H^{-g}}^2 = 0 \quad \mathbb{P} - a.s.$$

Thus, if we take a sequence  $\{t_k\}_{k=1}^\infty$  which is dense in  $[0, T]$  we have

$$\mathbb{P}\{\|V(t_k)\|_{H^{-g}} = 0 \text{ for all } k \in \mathbb{N}\} = 1.$$

Since each path of the process  $V$  belongs to  $C([0, T]; H^{-g})$ , we get (4.1).  $\square$

We recall that a strong solution of (2.14) is a progressively measurable process  $v : [0, T] \times \hat{\Omega} \rightarrow H$  fulfilling (2.15) for any given stochastic basis  $(\hat{\Omega}, \hat{\mathbb{F}}, \hat{\mathbb{P}})$  and any  $Y$ -cylindrical Wiener process  $\hat{w}$ .

Pathwise uniqueness and existence of martingale solutions imply existence of strong solution, see, e.g. Ikeda Watanabe monograph [18]. Therefore, Theorems 3.6 and 4.1 imply

**Theorem 4.2.** *Let  $d = 2$  and  $v_0 \in H$ ,  $f \in L^p(0, T; H^{-1})$  for some  $p > 2$ . If all three assumptions **(G1)**-**(G2)**-**(G3)** are satisfied, then there exists a unique strong solution of (2.14).*

## 5. APPENDIX

**5.1.  $\gamma$ -radonifying operators.** We refer to [12], [28] for the definition and main properties of  $\gamma$ -radonifying operators.

Let  $Y$  be a real separable Hilbert space and  $E$  a real separable Banach space; let  $\gamma_Y$  be the standard cylindrical Gaussian measure of  $Y$ . We denote by  $\mathcal{L}(Y; E)$  the space of linear bounded operators from  $Y$  to  $E$ , and by  $\gamma(Y; E)$  the space of  $\gamma$ -radonifying operators from  $Y$  to  $E$ .  $T \in \gamma(Y; E)$  means that  $T \in \mathcal{L}(Y; E)$  and  $T(\gamma_Y)$  extends to a Gaussian measure on  $E$ .

We recall the following well-known facts:

- If  $T : Y \rightarrow E$  is  $\gamma$ -radonifying and  $S : E \rightarrow F$  is bounded, then also  $S \circ T : Y \rightarrow F$  is  $\gamma$ -radonifying;
- If  $T : Y_1 \rightarrow E$  is  $\gamma$ -radonifying and  $S : Y_0 \rightarrow Y_1$  is bounded, then  $T \circ S : Y_0 \rightarrow E$  is  $\gamma$ -radonifying;
- If  $E$  is a Hilbert space, then  $T : Y \rightarrow E$  is  $\gamma$ -radonifying if and only if  $T$  is Hilbert-Schmidt.

We have the following characterization of  $\gamma$ -radonifying operators when  $E = L^p(\mathbb{R}^d)$ , see Proposition 13.7 in [28] and Theorem 2.3 in [12].

**Proposition 5.1.** *Let  $1 \leq p < \infty$  and  $\{e_j\}_{j=1}^\infty$  a complete orthonormal system in  $Y$ . For an operator  $T \in \mathcal{L}(Y, L^p(\mathbb{R}^d))$  the following assertions are equivalent:*

- (1)  $T \in \gamma(Y, L^p(\mathbb{R}^d))$ ;
- (2)  $(\sum_{j=1}^\infty |Te_j|^2)^{\frac{1}{2}} \in L^p(\mathbb{R}^d)$ .

Moreover the norms  $\|T\|_{\gamma(Y, L^p(\mathbb{R}^d))}$  and  $\|(\sum_j |Te_j|^2)^{\frac{1}{2}}\|_{L^p(\mathbb{R}^d)}$  are equivalent.

For the stochastic integral

$$X(t) = \int_0^t \phi(s) dW(s)$$

of a progressively measurable process  $\phi : [0, T] \times \Omega \rightarrow \gamma(Y; E)$  with respect to a  $Y$ -cylindrical Wiener process  $W$ , we have the following result, see e.g. [3].

**Proposition 5.2.** *Let  $W$  be a  $Y$ -cylindrical Wiener process.*

*If for some  $m \geq 2$  we have  $\mathbb{E}[(\int_0^T \|\phi(t)\|_{\gamma(Y; E)}^2 dt)^{m/2}] < \infty$ , then  $X$  has a progressively measurable  $E$ -valued version and*

$$\mathbb{E}\|X(t)\|_E^m \leq C_m \mathbb{E} \left[ \left( \int_0^t \|\phi(s)\|_{\gamma(Y; E)}^2 ds \right)^{\frac{m}{2}} \right].$$

**5.2. Compactness lemmas.** In [25, 7] there are some useful compactness results. We follow [7].

Given  $1 < p < \infty$ , we denote by  $L_w^p(0, T; L^4)$  the space  $L^p(0, T; L^4)$  with the weak topology.

For any  $R > 0$  let  $B_R = \{x \in \mathbb{R}^d : |x| < R\}$ . Given  $v : \mathbb{R}^d \rightarrow \mathbb{R}^d$ , we denote by  $v|_{B_R}$  its restriction to the ball  $B_R$ . We define

$$H_R = \{v|_{B_R} : v \in H\}, \quad H_R^\delta = \{v|_{B_R} : v \in H^\delta\}$$

with

$$\|v\|_{H_R} = \left( \int_{B_R} |v(x)|^2 dx \right)^{1/2}, \quad \|v\|_{H_R^\delta} = \left( \int_{B_R} |(\Lambda^\delta v)(x)|^2 dx \right)^{1/2}.$$

For  $\delta > 0$  the space  $H_R^\delta$  is compactly embedded in  $H_R$ , since the space variable belongs to a bounded set.

We denote by  $L^2(0, T; H_{\text{loc}})$  the space of measurable functions  $v : [0, T] \rightarrow H$  such that for any  $R > 0$  the norm

$$\|v\|_{L^2(0, T; H_R)} = \left( \int_0^T \int_{B_R} |v(t, x)|^2 dx dt \right)^{1/2}$$

is finite. It is a Fréchet space with the topology generated by the seminorms  $\|v\|_{L^2(0, T; H_R)}$ ,  $R \in \mathbb{N}$ .

**Lemma 5.3.** *Let*

$$\tilde{Z} = L_w^p(0, T; L^4) \cap C([0, T]; U') \cap L^2(0, T; H_{\text{loc}})$$

for some  $p \in (1, \infty)$ , and let  $\tilde{\mathcal{T}}$  be the supremum of the corresponding topologies. Then a set  $\tilde{K} \subset \tilde{Z}$  is  $\tilde{\mathcal{T}}$ -relatively compact if the following conditions hold:

- (i)  $\sup_{v \in \tilde{K}} \|v\|_{L^p(0, T; L^4)} < \infty$
- (ii)  $\exists \gamma > 0 : \sup_{v \in \tilde{K}} \|v\|_{C^\gamma([0, T]; H^{-1})} < \infty$
- (iii)  $\exists \delta > 0 : \sup_{v \in \tilde{K}} \|v\|_{L^2(0, T; H^\delta)} < \infty$ .

*Proof.* We can assume that  $\tilde{K}$  is closed in  $\tilde{\mathcal{T}}$ .

It is trivial from (i) that the set  $\tilde{K}$  is compact in  $L_w^p(0, T; L^4)$ ; this comes from the Banach-Alaoglu theorem. Moreover (ii) implies that the functions  $v \in \tilde{K}$  are equicontinuous, i.e.

$$\forall \varepsilon > 0 \exists \delta > 0 : |t - s| < \delta \implies \|v(t) - v(s)\|_{H^{-1}} \leq \varepsilon \quad \forall v \in \tilde{K}.$$

Since  $H^{-1}$  is compactly embedded in  $U'$ , Ascoli-Arzelà theorem provides that  $\tilde{K}$  is compactly embedded in  $C([0, T]; U')$ .

Notice that the compactness of a subset of  $\tilde{Z}$  is equivalent to its sequential compactness. Therefore, if we take a sequence  $\{v_i\}_{i \in \mathbb{N}} \subset \tilde{K}$  then there exists a subsequence converging to some  $v \in \tilde{K}$  in the two previous topologies. What remains to prove is that this subsequence (or possibly a subsubsequence) is convergent in  $L^2(0, T; H_{\text{loc}})$ .

Let us fix  $R > 0$ ; the embedding  $H_R^\delta \subset H_R$  is compact and  $H_R \simeq H'_R \subset H' \subset U'$  with continuous embeddings. Hence by a Lions Lemma, see [22], for every  $\varepsilon > 0$  there exists a constant  $C = C_{\varepsilon, R}$  such that

$$\|v\|_{H_R}^2 \leq \varepsilon \|v\|_{H_R^\delta}^2 + C \|v\|_{U'}^2.$$

Thus for almost all  $s \in [0, T]$

$$\|v_n(s) - v(s)\|_{H_R}^2 \leq \varepsilon \|v_n(s) - v(s)\|_{H_R^\delta}^2 + C \|v_n(s) - v(s)\|_{U'}^2,$$

and therefore, setting  $M = \sup_{v \in K} \|v\|_{L^2(0,T;H^\delta)}^2$  we get

$$\begin{aligned} \|v_n - v\|_{L^2(0,T;H_R)}^2 &\leq \varepsilon \|v_n - v\|_{L^2(0,T;H_R^\delta)}^2 + C \|v_n - v\|_{L^2(0,T;U')}^2 \\ &\leq \varepsilon 2M + CT \|v_n - v\|_{C([0,T];U')}^2 \end{aligned}$$

Now we pass to the upper limit (in a subsequence) and get

$$\limsup_{j \rightarrow \infty} \|v_{n_j} - v\|_{L^2(0,T;H_R)}^2 \leq 2M\varepsilon;$$

since  $\varepsilon$  is arbitrary, we get the convergence in  $L^2(0,T;H_{\text{loc}})$ . □

If we proceed as in Lemma 3.3 in [7] or Lemma 2.7 in [25], we get also the following result.

**Lemma 5.4.** *Let*

$$Z = L_w^p(0,T;L^4) \cap C([0,T];U') \cap L^2(0,T;H_{\text{loc}}) \cap C([0,T];H_w)$$

for some  $p \in (1, \infty)$ , and let  $\mathcal{T}$  be the supremum of the corresponding topologies. Then a set  $K \subset Z$  is  $\mathcal{T}$ -relatively compact if the following conditions hold:

- (i)  $\sup_{v \in K} \|v\|_{L^p(0,T;L^4)} < \infty$
- (ii)  $\exists \gamma > 0 : \sup_{v \in K} \|v\|_{C^\gamma([0,T];H^{-1})} < \infty$
- (iii)  $\exists \delta > 0 : \sup_{v \in K} \|v\|_{L^2(0,T;H^\delta)} < \infty$
- (iv)  $\sup_{v \in K} \|v\|_{L^\infty(0,T;H)} < \infty$ .

From this lemma we also get a tightness criterion. We recall that a family of probability measures  $\{P_n\}_n$ , defined on the  $\sigma$ -algebra of Borel subsets of  $Z$ , is tight if for any  $\varepsilon > 0$  there exists a compact subset  $K_\varepsilon$  of  $Z$  such that

$$\inf_n P_n(K_\varepsilon) \geq 1 - \varepsilon$$

or equivalently

$$\sup_n P_n(Z \setminus K_\varepsilon) \leq \varepsilon.$$

**Lemma 5.5** (tightness criterion). *We are given parameters  $\gamma > 0$ ,  $\delta > 0$ ,  $1 < p < \infty$  and a sequence  $\{v_n\}_{n \in \mathbb{N}}$  of adapted processes in  $C([0,T];U')$ .*

*Assume that for any  $\varepsilon > 0$  there exist positive constants  $R_i = R_i(\varepsilon)$  ( $i = 1, \dots, 4$ ) such that*

$$(5.1) \quad \sup_n \mathbb{P}(\|v_n\|_{L^p(0,T;L^4)} > R_1) \leq \varepsilon$$

$$(5.2) \quad \sup_n \mathbb{P}(\|v_n\|_{C^\gamma([0,T];H^{-1})} > R_2) \leq \varepsilon$$

$$(5.3) \quad \sup_n \mathbb{P}(\|v_n\|_{L^2(0,T;H^\delta)} > R_3) \leq \varepsilon$$

$$(5.4) \quad \sup_n \mathbb{P}(\|v_n\|_{L^\infty(0,T;H)} > R_4) \leq \varepsilon$$

*Let  $\mu_n$  be the law of  $v_n$  on  $Z = L_w^p(0,T;L^4) \cap C([0,T];U') \cap L^2(0,T;H_{\text{loc}}) \cap C([0,T];H_w)$ . Then, the sequence  $\{\mu_n\}_{n \in \mathbb{N}}$  is tight in  $Z$ .*

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